

More about the new non-selfdual axially symmetric $SU(2)$ calorons

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We describe the recently found non-selfdual axially symmetric caloron solutions of $SU(2)$ gluodynamics with trivial holonomy. We present the local Polyakov loop together with the action and topological charge density. Different from the well-known KvBLL calorons, the calorons considered here are composed out of pseudoparticle constituents carrying *integer* topological charge. The loci of Polyakov loop $L = -1$ correspond to the loci of Higgs field zeroes for corresponding solutions of the Yang-Mills Higgs system. For certain parameters pointlike monopole pairs turn into rings.

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1. Introduction

The relations between the properties of selfdual BPS monopole solutions [1] and instantons [2] attracted a lot of interest over the last decade. It was shown that exact caloron solutions, *i.e.* time-periodic instantons at finite temperature on $\mathbb{R}^3 \times S^1$, for which the temporal gauge field component A_0 approaches some constant at spatial infinity [3, 4], $A_0 \rightarrow 2\pi i\omega = 2\pi i\omega^a \sigma^a$, are composed out of Bogomol'nyi-Prasad-Sommerfeld (BPS) monopole (and antimonopole) constituents [5]. This time-periodic instanton eventually corresponds to a non-trivial Polyakov loop (holonomy around S^1) at spatial infinity. In the periodic gauge $A_\mu(\mathbf{r}, x_0 + T) = A_\mu(\mathbf{r}, x_0)$ the Polyakov loop is defined as

$$\mathcal{P}(\mathbf{r}) = \lim_{r \rightarrow \infty} P \exp \left(\int_0^T A_0(\mathbf{r}, x_0) dx_0 \right), \quad (1.1)$$

where T is the period in the imaginary time direction, which is related to the temperature Θ through $T = 1/\Theta$. The symbol P denotes path ordering.

The property of selfduality allows to apply the very powerful ADHM-Nahm formalism [7] to obtain various exact multi-caloron configurations [5, 8] and to analyse the properties of the BPS monopole constituents. In particular, it was shown that, if the size of a charge one $SU(2)$ caloron is getting larger than the period T , the caloron is splitting into constituents, *i.e.* the monopole-antimonopole pair becomes visible through well-separated lumps of action. The properties of the saddle point solutions in the related $SU(2)$ Yang-Mills-Higgs (YMH) model were discussed first by Taubes [9], and various monopole-antimonopole YMH systems were constructed numerically in Refs. [10, 11, 12], both in the BPS limit and beyond.

However, besides the selfdual instantons, also solutions of the second order Euler-Lagrange equations of the Euclidean Yang-Mills (YM) theory are known [13]. Thus, a non-selfdual instanton-antiinstanton pair static configuration has been constructed [14], which represents a saddle point configuration, *i.e.* a deformation of the topologically trivial field.

Recently new static and axially symmetric $SU(2)$ YM caloron solutions on $\mathbb{R}^3 \times S^1$ with trivial holonomy were constructed numerically [15]. These regular field configurations are labeled by two integers n and m , analogously to their counterparts in the YMH system, the monopole-antimonopole chains and the circular vortices [12]. Similar to the case of axially symmetric instantons discussed in [14], only the $m = 1$ solutions are selfdual. The calorons labeled by $m \geq 2$ are non-selfdual. The latter are also composed of constituents and correspond to the monopole-antimonopole chains and/or to the vortex-like solutions of Ref. [15].

In this talk we briefly describe the properties of the new non-selfdual, axially symmetric caloron solutions of the second order field equations, adding here the results of the numerical evaluation of the holonomy in the caloron background to the main characteristics.

2. The Euclidean $SU(2)$ action and the axially symmetric ansatz

We consider the usual $SU(2)$ YM action

$$S = \frac{1}{2} \int d^4x \text{Tr} (F_{\mu\nu} F_{\mu\nu}) = \frac{1}{4} \int d^4x \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu} \right)^2 \mp \frac{1}{2} \int d^4x \text{Tr} (F_{\mu\nu} \tilde{F}_{\mu\nu}). \quad (2.1)$$

We work in Euclidean space $R^3 \times S^1$ with compactified time running in $x_0 \in [0, T]$. The gauge coupling is put equal to $e^2 = 1$. The $su(2)$ gauge potential is $A_\mu = A_\mu^a \tau^a / 2$, and the field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. The topological charge $Q = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \int d^4x \text{Tr} F_{\mu\nu} F_{\rho\sigma}$ is defined by the integral over all space-time. Only for selfdual configurations the bound for the action $S \geq 8\pi^2 |Q|$ (and a similar local bound for the densities) is saturated.

To construct the new regular caloron solutions of the corresponding *second order* field equations and in order to investigate the dependence of these solutions on the boundary conditions, we employ the axially symmetric ansatz for the gauge field

$$A_\mu dx^\mu = \left(\frac{K_1}{r} dr + (1 - K_2) d\theta \right) \frac{\tau_\phi^{(n)}}{2e} - n \sin \theta \left(K_3 \frac{\tau_r^{(n,m)}}{2e} + (1 - K_4) \frac{\tau_\theta^{(n,m)}}{2e} \right) d\phi;$$

$$A_0 = A_0^a \frac{\tau^a}{2} = \left(K_5 \frac{\tau_r^{(n,m)}}{2} + K_6 \frac{\tau_\theta^{(n,m)}}{2} \right), \quad (2.2)$$

that was previously applied to the YMH system [12]. The ansatz is written in the basis of $su(2)$ matrices $\tau_r^{(n,m)}$, $\tau_\theta^{(n,m)}$ and $\tau_\phi^{(n)}$ which are defined as the dot product between the Cartesian vector of Pauli matrices $\vec{\tau}$ and the spatial unit vector that generalizes the local dreibein $\vec{x}_i/|\vec{x}_i| = \hat{e}_i^{(1,1)}$ of tangential vectors ($i = r, \theta, \phi$ denote the tangential directions). The generalization proceeds analogously to the radial one, $\hat{e}_r^{(1,1)} \rightarrow \hat{e}_r^{(n,m)} = (\sin(m\theta) \cos(n\phi), \sin(m\theta) \sin(n\phi), \cos(m\theta))$, by changing $\phi \rightarrow n\phi$ and $\theta \rightarrow m\theta$. The gauge field functions $K_i(r, \theta)$ ($i = 1, \dots, 6$) depend only on the spherical coordinates r and θ .

Substitution of the axially symmetric ansatz (2.2) into the definition of the topological charge Q yields $Q = \frac{n}{2} [1 - (-1)^m]$, similar to [12, 14]. Thus, the configurations labeled by even m belong to the topologically trivial sector and represent saddle point solutions.

To satisfy the condition of finiteness of the total Euclidean action (2.1), we additionally require that the field strength vanishes as $\text{Tr}(F_{\mu\nu} F_{\mu\nu}) \rightarrow O(r^{-4})$ with $r \rightarrow \infty$. In the regular gauge the value of the A_0 component of the gauge potential approaches a constant at spatial infinity, *i.e.* $A_0 \rightarrow \frac{i\beta}{2} \tau_r^{(n,m)}$. This corresponds to the holonomy at infinity (1.1),

$$\frac{1}{2} \text{Tr} \mathcal{P}(\vec{r}) \rightarrow \frac{1}{2} \text{Tr} \exp \left(\frac{i\beta T}{2} \tau_r^{(n,m)} \right) = \frac{1}{2} \text{Tr} U \exp \left(\frac{i\beta T}{2} \tau_z \right) U^{-1} = \cos \frac{\beta T}{2}, \quad (2.3)$$

where $U \in SU(2)$ and $\beta \in [0; 2\pi/T]$.

We consider now deformations in the topologically trivial sector and deformations of the caloron solution with trivial holonomy at infinity [3, 16]. The latter is defined as a time-periodic array of $Q = 1$ or $Q = -1$ instantons, located along the Euclidean time axis (with distance T). A possible generalization of this solution corresponds to a time-periodic array of instantons with general charge Q [16]. Now we will not require anymore that the gauge field should be selfdual such that, generically, $F_{\mu\nu}(x) \neq \pm \tilde{F}_{\mu\nu}(x)$.

The regular caloron solutions with finite action density and proper asymptotic behavior can be constructed numerically by imposing boundary conditions [15] and solving the resulting system of six coupled non-linear partial differential equations of second order [15]. As usual, to obtain a regular solutions we have to satisfy the gauge condition $\partial_r A_r + \partial_\theta A_\theta = 0$ [12]. We also introduce a compact radial coordinate $x = r/(1+r) \in [0; 1]$. The numerical calculations were performed with the software package FIDISOL based on the Newton-Raphson iterative procedure.

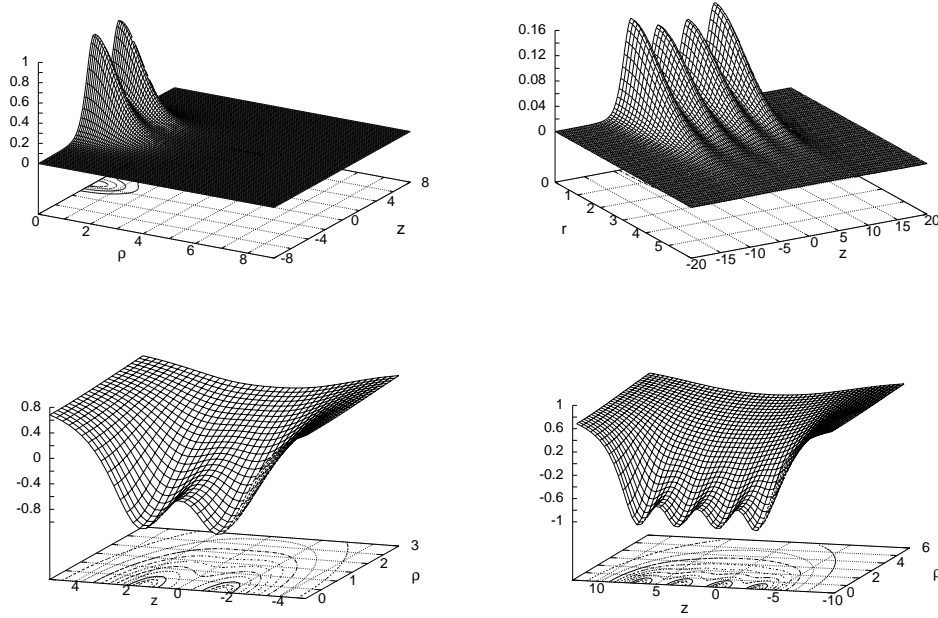


Figure 1: The action density for the $m = 2$ (upper left) and $m = 4$ (upper right) caloron chains, both with winding number $n = 1$, is shown vs. cylindrical coordinates z and ρ . The Polyakov loop is plotted below.

3. Discussion of the solutions

The simplest class of solutions corresponds to $m = 1$. It turns out that, similar to [14], these solutions are selfdual. We check this conclusion by numerical calculation of the integrated action density, as well as direct substitution of the solutions into the first order equation expressing selfduality. Furthermore, the $m = n = 1$ solution is nothing but the Harrington-Shepard [3] spherically symmetric finite temperature solution of unit topological charge. The solutions with $m = 1$ and $n \geq 2$ are of reduced, *i.e.* axial symmetry. Their distribution of action has the shape of a torus around the z -axis.

The $m \geq 2$ configurations do not satisfy the first order (selfduality) equations. Similarly to their counterparts in YMH theory [12], the solutions with $n = 1$ and $m = 2, 3, 4, \dots$ represent time-periodic arrays of finite-length chains containing instantons and anti-instantons of $Q = \pm 1$ topological charge in alternating order, located along the spatial symmetry axis with m clearly separated maxima of the action density (see Fig 1). The topological charge density possesses m local extrema along the z axis, whose locations coincide with the maxima of the action density. Thus, one can distinguish m individual constituents.

To compute the local Polyakov loop at a given point \vec{x} we note that in the static regular gauge the temporal component A_0 of the gauge potential approaches a constant at spatial infinity although the holonomy is trivial. To check it, we make use of the numerical solutions found above and perform the integration in the exponent (in the static gauge). One gets $\text{Tr } \mathcal{P}(\vec{r}) = \cos ||A_0(\vec{r})||$ which agrees with (2.3) asymptotically.

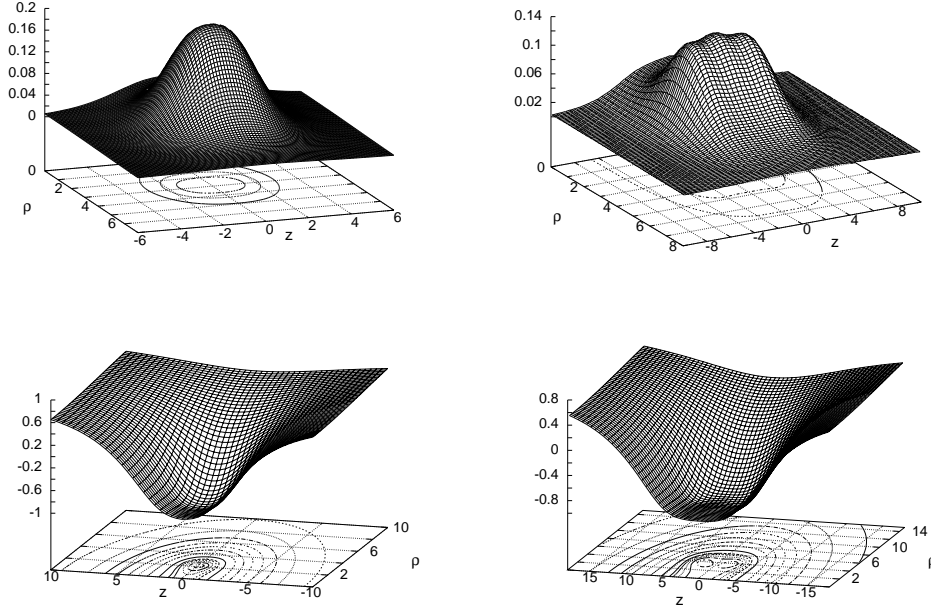


Figure 2: The action density plotted for the $n = 3, m = 2$ (upper left) and $n = 3, m = 3$ (upper right) caloron chain solutions. The Polyakov loop is plotted for the $n = 3, m = 2$ (bottom left) caloron and compared with the $n = 4, m = 4$ (bottom right) solution.

For the solutions of the chain type with $n = 1$ and $m = 2, 3, 4, \dots$ the Polyakov loop takes extremal values $\mathcal{P}(\vec{x}_i) = -1$ (opposite to the asymptotic value $\mathcal{P}(|\vec{x}| \rightarrow \infty) = 1$) on the symmetry axis where the constituents are located at $\vec{x}_i = (0, 0, z_i)$ (see Fig. 1). In this sense the local Polyakov loop perfectly corresponds to the action density.

The same general behavior is observed for the other solutions. Generally, the winding number n is related to the (integer) topological charge of each individual pseudoparticle. Increasing n to $n > 1$ deforms the local maxima of the action density into tori around the symmetry axis with a nonvanishing (and increasing with n) radius. For example, a single ring is formed for the configuration with $n = 3$ and $m = 2$ (see the upper plots of Fig. 2). Similarly, the local Polyakov loop now takes on the value $\mathcal{P}(\vec{x}) = -1$ on circles in the core of the torus around the symmetry axis with radius ρ_n (see the bottom plots of Fig. 2) where the action density has a toroidal maximum. For a $Q = 3$ instanton-antiinstanton configuration with $n = m = 3$ we found three maxima of the action density located in three $x-y$ -planes corresponding to three tori, one sitting at $z = 0$ and the other two sitting symmetrically at $z = \pm \Delta z$ (see the right upper plot of Fig. 2). The local holonomy precisely mimicks this behavior.

4. Conclusions

We have constructed axially symmetric caloron solutions for the four-dimensional Euclidean $SU(2)$ YM theory by numerical solution of the second order Yang-Mills equations with trivial

asymptotic holonomy. Similarly to the axially symmetric monopole-antimonopole and vortex ring solutions of the YMH theory, the calorons are labeled by two winding numbers, n and m . They are consisting of pseudoparticles of topological charge $\pm n$ building up a total topological charge $Q = \frac{n}{2} [1 - (-1)^m]$. The action density of the configuration has a non-trivial shape. The position of its maxima allow us to identify a pointlike location or a toroidal shape of each individual constituent, depending on the value of n . We have studied here also the landscape of the holonomy or Polyakov loop. In all cases the loci of $\mathcal{P}(\mathbf{r}_0) = -1$ coincide with the maxima of the action density.

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